Annuities and Sinking Funds

Finite Math

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Annuities

At this point, we have only discussed investments where there was one initial deposit and a final payoff. But what if you make regular equal payments into an account? An *annuity* is a sequence of equal periodic payments. If payments are made at the end of each time interval, then the annuity if called an *ordinary annuity*. Our goal will be to find the future value of an annuity.

Future Value of an Annuity

Example

Suppose you decide to deposit \$100 *every* 6 *months into a savings account which pays* 6% *compounded semiannually. If you make* 6 *deposits, one at the end of each interest payment period over the course of* 3 *years, how much money will be in the account after the last deposit is made?*

Solution

We can visualize the value of each of those $100 deposits in a table.

Deposit	Term	# of times Compounded	Future Value
$100	1	5	$\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$
$100	2	4	$\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$
$100	3	3	$\$100\left(1+\frac{0.06}{2}\right)^3 = \$100(1.03)^3$
$100	4	2	$\$100\left(1+\frac{0.06}{2}\right)^2 = \$100(1.03)^2$
$100	5	1	$\$100\left(1+\frac{0.06}{2}\right)^1 = \$100(1.03)$
$100	6	0	$\$100\left(1+\frac{0.06}{2}\right)^0 = \$100$

So adding up the future values of all these will give us the amount of money in the account

$$\begin{aligned} B &= \$100(1.03)^5 + \$100(1.03)^4 + \$100(1.03)^3 \\ &\quad + \$100(1.03)^2 + \$100(1.03) + \$100 = \$646.84 \end{aligned}$$

Future Value

Definition (Future Value of an Ordinary Annuity)

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^n - 1}{r/m}$$

where

$$\begin{aligned} FV &= \text{future value} \\ PMT &= \text{periodic payment} \\ m &= \text{frequency of payments} \\ n &= \text{number of payments (periods)} \end{aligned}$$

Note that the payments are made at the end of each period.

Future Value

Example

What is the value of an annuity at the end of 10 *years if* $\$1,000$ *is deposited every* 3 *months into an account earning* 8% *compounded quarterly. How much of this value is interest?*

Now You Try It!

Example

If $\$1,000$ *is deposited at the end of each year for* 5 *years into an ordinary annuity earning* 8.32% *compounded annually, what will be the value of the annuity at the end of the* 5 *years?*

Solution

$\$5,904.15$

Sinking Funds

We can turn the annuities picture around and ask how much we would need to deposit into an account each period in order to get the desired final value. It is simple to solve for *PMT* in the annuities formula to get

Definition (Sinking Funds)

$$PMT = FV \frac{r/m}{\left(1 + \frac{r}{m}\right)^n - 1}$$

where all the variables have the same meaning as for annuities.

Sinking Funds

Example

New parents are trying to save for their child's college and want to save up $\$80,000$ *in* 17 *years. They have found an account that will pay* 8% *interest compounded quarterly. How much will they have to deposit every quarter in order to have a value of* $\$80,000$*?*

Now You Try It!

Example

A bond issue is approved for building a marina in a city. The city is required to make regular payments every 3 *months into a sinking fund paying* 5.4% *compounded quarterly. At the end of* 10 *years, the bond obligation will be retired with a cost of* $\$5,000,000$*. How much will the city have to pay each quarter?*

Solution

$\$95,094.67$